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Multi-Grid Radiative Transfer Revisited

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Abstract. Multi-grid radiative transfer is an efficient solution method for a variety of radiative transfer problems, in particular for problems of multiple spatial dimensions. Contrary to the commonly used operator splitting methods, the convergence rate of the multi-grid method does not deteriorate with increasing spatial resolution. It should therefore be well suited for high resolution problems.

1. What Is Multi-Grid Radiative Transfer and How Does It Work?

The multi-grid method is an efficient algorithm for accelerating convergence of iterative methods. The scheme alternates between an iteration that quickly reduces local (high frequency) errors, and the approximate solution on a coarse grid for reducing the global (low frequency) error. *Multi-grid radiative transfer* is the application of this scheme to radiative transfer. A good monograph on the foundations of multi-grid methods is given by Hackbusch (1985), a comprehensive web-site on multi-grid methods is maintained by Craig C. Douglas under `www.mgnet.org`. This paper reviews the basic method and some results of multi-grid radiative transfer.

Consider the source function, $S(\tau) = (1-\epsilon)J(\tau) + \epsilon B$, with ϵ being the thermalization parameter and J the mean intensity, $J(\tau) = \mathbf{\Lambda}(\tau, t)S(t)$, where $\mathbf{\Lambda}(\tau, t)$ denotes the formal integration of the radiative transfer equation. B is the Planck function. Substitution leads to the *common* $\mathbf{\Lambda}$*-iteration*

$$S^{(n+1)} = (1-\epsilon)\mathbf{\Lambda} S^{(n)} + \epsilon B \ . \tag{1}$$

Better convergence is achieved with the *"accelerated* $\mathbf{\Lambda}$*-iteration" (ALI)*:

$$S^{(n+1)}(\tau) = [\mathbf{I} - (1-\epsilon)\mathbf{\Lambda}^*]^{-1}(1-\epsilon)(\mathbf{\Lambda} - \mathbf{\Lambda}^*)S^{(n)}(\tau) + \epsilon B \ , \tag{2}$$

where $\mathbf{\Lambda}^*$ is a suitable approximation to $\mathbf{\Lambda}$.

Iterations of the type of Eq. (2) are efficient in reducing the high frequency components of the error of an intermediate solution, $S^{(n)}(\tau)$, (it makes the error smooth) but poorly converge with respect to the low frequency components (the overall error). The idea of the multi-grid method is to approximate this remaining smooth error on a coarse grid at accordingly low computational cost and with radiative coupling of increased scales.

Given an approximation, $S_l^{(0)}$, to the exact solution, S_l, on a finest level l grid. A small number (usually only one) of accelerated $\mathbf{\Lambda}$-iterations applied on

$S_l^{(0)}$, the *smoothing step*, results in the intermediate solution, $\bar{S}_l$. $v_l := \bar{S}_l - S_l$ is smoother than the original error, $S_l^{(0)} - S_l$. Defining the *defect*, $d_l := \bar{S}_l - (1-\epsilon)\mathbf{\Lambda}_l \bar{S}_l - \epsilon B_l$, which is the difference between two subsequent common $\mathbf{\Lambda}$-iterations, we can compute the error, v_l, as:

$$v_l = (1-\epsilon)\mathbf{\Lambda}_l v_l + d_l \; . \tag{3}$$

Equation (3) is of exactly the same form as the original problem, Eq. (1). But since v_l is a smooth grid function it may well be represented on a coarse grid of level $l-1$ with half the number of grid points of the level l grid:

$$v_{l-1} = (1-\epsilon)\mathbf{\Lambda}_{l-1} v_{l-1} + d_{l-1} \; , \tag{4}$$

which is called the *coarse-grid equation.* Having computed the error on the coarse grid we *prolongate* it back to the fine grid, thus obtaining the *coarse grid correction*, $\tilde{v}_l$, for a new approximation to the source function: $S_l^{\mathrm{new}} = \bar{S}_l - \tilde{v}_l$. Since the coarse-grid equation, Eq. (4), is of the same form as the original problem, Eq. (1), the same two-grid algorithm, described so far, can be recursively applied to Eq. (4), leading to a multi-level algorithm.

2. Applications

2.1. Homogeneous Slab of Two-Level Atoms

Figure 1 shows iterative approximations to the source function of a homogeneous slab of optical thickness $\tau = 2000$ and a thermalization parameter $\epsilon = 10^{-6}$. One decade in τ contains 6 grid points. Fig. 2a shows subsequent approximations of an accelerated $\mathbf{\Lambda}$-iteration with a non-local $\mathbf{\Lambda}^*$ (nearest neighbour coupling), which have by far not converged as S/B should be $\sqrt{\epsilon} = 10^{-3}$ at the edge of the slab. Fig. 2b shows that after only two multi-grid cycles the approximation is very close to the final solution.

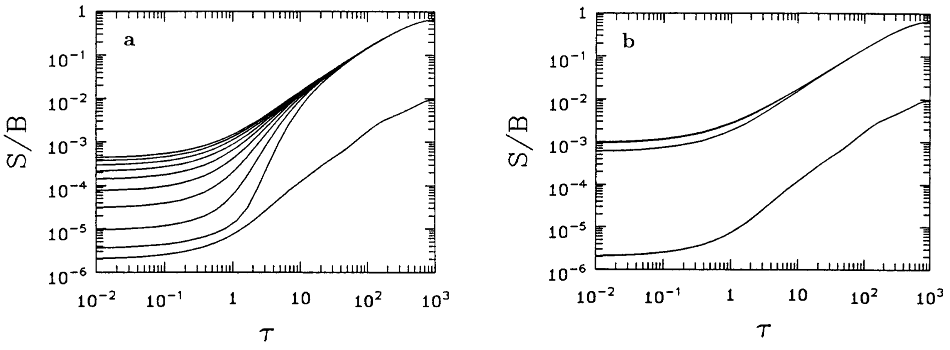

Figure 1. Subsequent approximations of the source function with **a** the non-local accelerated $\mathbf{\Lambda}$-iteration, **b** the multi-grid method with 3 levels. (From Steiner 1991)

2.2. Atmosphere in Radiative Equilibrium (RE)

The multi-grid scheme was also applied to the case of radiative transfer under the constraint of radiative equilibrium, $B(\mathbf{r}) = J(\mathbf{r})$, with

$$J(\mathbf{r}) = \mathbf{\Lambda}(\mathbf{r}, \mathbf{r}')B(\mathbf{r}') + G(\mathbf{r}) \; . \tag{5}$$

$\mathbf{r}$ is the location in a two or three-dimensional coordinate frame. G is the incident radiation from the boundaries.

Steiner (1991) considers a two-dimensional RE-problem and found the multi-grid method to be 40 times faster (in wall-clock time) than ALI. Väth (1994) applies the multi-grid method to problems of three-dimensional radiative transfer. He uses a massively parallel SIMD computer with 8192 processors.

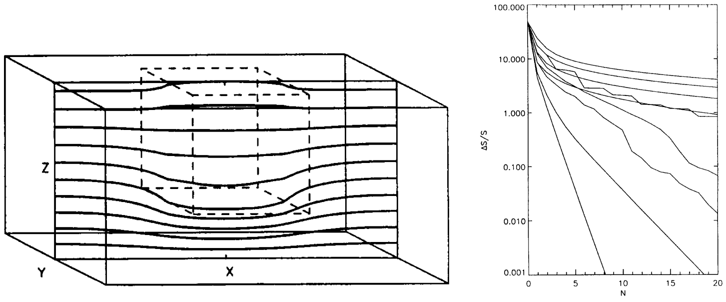

Figure 2. **Left**: Isotherms in the mid-plane of the 3-D atmosphere in RE. **Right:** Maximum relative error of the source function as function of iteration steps. From top to bottom (worst to best): $\mathbf{\Lambda}$-iteration, local ALI, non-local ALI, 2 level MGM, Ng acceleration, local orthomin, non-local orthomin, level 3 MGM, level 4 MGM. (From Väth 1994)

Figure 2 (left) shows the geometry and resulting isotherms. The opacity within the dashed cube is reduced to 1/5 of that in the ambient medium. Boundaries are periodic in x and y-direction. Isotropic radiation is assumed at the bottom, while no radiation is incident on the top.

Figure 2 (right) shows that the multi-grid methods with 3 and 4 levels converge smooth and fastest. However, since the computational costs of one multi-grid iteration is 2 to 7 times that of a single $\mathbf{\Lambda}$-iteration, the orthomin method is more efficient. This is because with massively parallel computation the CPU-time needed for a formal solution scales with N, compared with N^3 in scalar mode. Also load balance is difficult to maintain. Väth (1994) concludes that multi-grid radiative transfer is not well suited for massively parallel computing.

2.3. Non-Linear Multigrid Radiative Transfer for Multilevel Atoms in Multi-Dimensional Space

Fabiani Bendicho, Trujillo Bueno, & Auer (1997) apply multi-grid radiative transfer to the multi-level non-LTE problem in one and two spatial dimensions. They use a Gauss-Seidel iteration scheme (MUGA) for solving the non-linear population equation and emphasize the excellent smoothing capability of

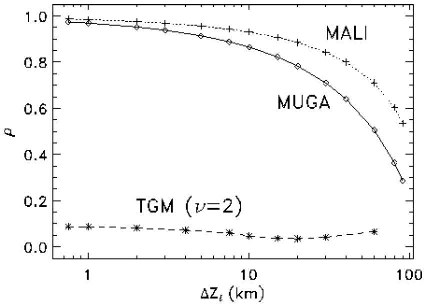

Figure 3. Contraction number for the multilevel accelerated $\mathbf{\Lambda}$-iteration (MALI), the multilevel Gauss-Seidel method (MUGA), and for the two-grid method (TGM).

MUGA, making it an ideal choice for the smoothing iterations in the multi-grid method.

The *non-linear multi-grid method* differs from the linear case in that the coarse-grid equation solves for the occupation numbers, $\mathbf{n}_{l-1}$, directly and not just for the correction $\Delta\mathbf{n}_{l-1}$, and in that the restriction operator is not only applied to the residual but to the current estimate, $\mathbf{n}_l^{\mathrm{old}}$, as well.

Figure 3 shows the contraction number as a function of grid spacing (in km) of three schemes for the test case of a 5-level Ca II atom in a one-dimensional (solar) atmosphere. The contraction number is basically the factor by which the error is reduced per iteration. The figure demonstrates the known and attractive property of multi-grid methods that their convergence rates are virtually independent on grid resolution, contrary to operator splitting methods that deteriorate when resolution is increased.

Balsara (2001) uses the non-linear multigrid method in conjunction with a flux conservative discretization of the radiative transfer equation in multiple spatial dimensions.

3. Concluding Remarks

There exists a variety of multi-grid algorithms, which might allow for further improvements of the few multi-grid radiative transfer calculations performed so far. The following bibliography is believed to be complete with respect to sources dealing with multi-grid radiative transfer.

References

Balsara, D. S. 2001, J. Quant. Spectrosc. Radiat. Transfer, 69, 671

Douglas C.C., `www.mgnet.org`.

Fabiani Bendicho, P., Trujillo Bueno, J., & Auer, L. 1997, 324, 161

Hackbusch, W 1985, Multi-Grid Methods and Applications, (Berlin: Springer)

Steiner, O. 1991, Astron. & Astroph., 242, 290

Väth, H. M. 1994, Astron. & Astroph., 284, 319