Multi–Grid Radiation Transfer Revisited

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What is Multi-grid Radiation Transfer ?

The multi-grid method is an efficient scheme for accelerating convergence of iterative methods. The scheme alternates between regular iterations by which local (high frequency) errors are quickly reduced and the approximate solution on a coarse grid for reducing the global (low frequency) error. *Multi-grid radiation transfer* is the application of this scheme to radiation transfer.

There exists an extensive literature on multi-grid methods. A good monograph on the foundations of multi-grid methods is by Hackbusch (1985). This poster reviews the basic method and some results of multi-grid radiation transfer.

How it works

Consider the source function

$$S(\tau) = (1-\epsilon)J(\tau) + \epsilon B\,. \tag{1}$$

ϵ is the thermalization parameter and J the mean intensity

$$J(\tau) = \Lambda(\tau, t)S(t)\,, \tag{2}$$

where $\Lambda(\tau, t)$ denotes the formal integration of the radiation transfer equation. Substitution of Eq. (1) into Eq. (2) leads to the common *Λ-iteration*

$$S^{(n+1)} = (1-\epsilon)\Lambda S^{(n)} + \epsilon B\,. \tag{3}$$

Better convergence is achieved with the *accelerated Λ-iteration*:

$$S^{(n+1)}(\tau) = [\mathbf{I} - (1-\epsilon)\Lambda^*]^{-1}(1-\epsilon)(\Lambda - \Lambda^*)S^{(n)}(\tau) + \epsilon B\,, \tag{4}$$

where Λ^* is a suitable approximation to Λ.
Iterations of the type of Eq. (4) are efficient in reducing the high frequency components of the error of an intermediate solution, $S^{(n)}(\tau)$, (It makes the error smooth!) but poorly converge with respect to the low frequency components. The *idea of the multi-grid method* is that this remaining smooth error can well be approximated on a coarse grid where it can be computed at low costs.

Consider an approximation $S_l^{(0)}$ to the exact solution S_l given on a finest level l grid. A small number (usually only one) of accelerated Λ-iterations applied on $S_l^{(0)}$, the *smoothing step*, results in the intermediate solution $\tilde{S}_l$.

$$v_l = \tilde{S}_l - S_l \tag{5}$$

is smoother than the original error $S_l^{(0)} - S_l$. Defining the *defect*

$$d_l := \tilde{S}_l - (1-\epsilon)\Lambda_l\tilde{S}_l - \epsilon B_l\,, \tag{6}$$

which is the difference between two subsequent common Λ-iterations, we can compute the error v_l as:

$$v_l = (1-\epsilon)\Lambda_l v_l + d_l\,. \tag{7}$$

Here we make use of Λ being a linear operator, which restriction can be removed in the non-linear multi-grid method. Note that Eq. (7) is of exactly the same form as the original problem, Eq. (3).

However, since v_l is a smooth grid function it may well be represented on a coarse grid of level $l-1$ with half the number of grid points of the fine grid of level l:

$$v_{l-1} = (1-\epsilon)\Lambda_{l-1}v_{l-1} + d_{l-1}\,. \tag{8}$$

Equation (8) is called the *coarse-grid equation*. d_{l-1} is computed by applying a *restriction* on d_l:

$$d_{l-1} = r d_l\,. \tag{9}$$

Having computed the error on the coarse grid we *prolongate* it back on the fine grid by piecewise quadratic or cubic interpolation

$$\tilde{v}_l = p v_{l-1}\,. \tag{10}$$

Thus, we obtain the *coarse grid correction* for a new approximation to the source function:

$$S_l^{\mathrm{new}} = \tilde{S}_l - \tilde{v}_l\,. \tag{11}$$

Since the coarse-grid equation, Eq. (8), is of the same form as the original problem, Eq. (3), the same two-grid algorithm, described so far, can be recursively applied to Eq. (8), leading to the following *multi-grid algorithm*:

procedure $MGM(l, S, f, sm)$;
  integer l; **array** S, f; **boolean** sm;
  if $l = 1$ **then** $S := (\mathbf{I} - (1-\epsilon)\Lambda_1)^{-1} f$ **else**
  begin array v, d;
    if sm **then** $S := (\mathbf{I} - (1-\epsilon)\Lambda_l^*)^{-1}$
        $[(1-\epsilon)(\Lambda_l - \Lambda_l^*)S + f]$;
    $d := r(S - (1-\epsilon)\Lambda_l S - f)$;
    $v := 0$;
    $MGM(l-1, v, d, false)$; $MGM(l-1, v, d, true)$;
    $S := S - pv$;
  end;

The course of the algorithm can be seen from the following diagram.

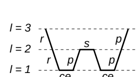

Fig. 1: Diagram of algorithm MGM (W-cycle) with three grid levels. r: restriction, ce: solution of the coarse-grid equation, p: prolongation, s: smoothing, l: grid level.

Note that only defect and correction are restricted and prolongated, respectively, not the solution. Algorithm MGM is called a W-cycle. Other cycle "shapes" that may be more efficient are possible by slight modifications of MGM.

Homogeneous slab

Figure 2 shows iterative approximations to the source function for a homogeneous slab of optical thickness $\tau = 2000$ and a thermalization parameter $\epsilon = 10^{-6}$. One decade in τ contains 6 grid points. Fig. 2a shows subsequent approximations of an accelerated Λ-iteration with a non-local Λ^* (nearest neighbour coupling). The sequence has by far not converged as S/B should be $\sqrt{\epsilon} = 10^{-3}$ at the edge of the slab. Fig. 2b shows that after only two W-cycles with algorithm MGM the approximation is very close to the final solution.

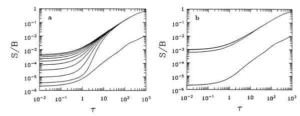

Fig. 2: See text for details.

Radiative equilibrium constraint

The multi-grid scheme can also be applied in the case of radiation transfer under the constraint of radiative equilibrium

$$B(\mathbf{r}) = J(\mathbf{r}) \tag{12}$$

with

$$J(\mathbf{r}) = \Lambda(\mathbf{r}, \mathbf{r}')B(\mathbf{r}') + G(\mathbf{r})\,. \tag{13}$$

$\mathbf{r}$ is the location in a two or three-dimensional coordinate frame. G is due to the incident radiation from the boundaries.

Figure 3 shows on the left hand side the isotherms of a two-dimensional atmosphere in radiative equilibrium, where the opacity is reduced to 1/5 of the surroundings within the rectangle in the upper left. The z-axis is an axis of symmetry.

The panel to the right of Fig. 3 compares the convergence for this problem of the accelerated Λ-iteration ($\cdot$) with the multi-grid method ($*$). The two scales account for a W-cycle consuming 2.2 times more CPU-time than an accelerated Λ-iteration. Still, the multi-grid method is *40 times faster* than ALI for this problem.

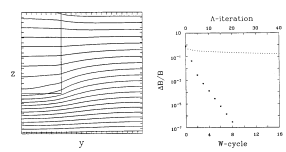

Fig. 3: See text for details.

Massively parallel multi-grid RT in 3-D

Väth (1994) applies the multi-grid method to problems of three-dimensional radiation transfer. He uses a massively parallel SIMD machine with 8192 processors. The two problems he considers are an extension of the problems shown in Figs. 2 and 3 to three dimensions.

Figure 4 shows isotherms in the mid-plane of the 3-D atmosphere in radiative equilibrium. The opacity within the dashed cube is reduced to 1/5 of that in the ambient medium. Boundaries are periodic in x and y-direction. Isotropic radiation is assumed at the bottom, while no radiation is incident on the top boundary from outside. Convergence for the solution of this problem with various methods are shown in Fig. 5.

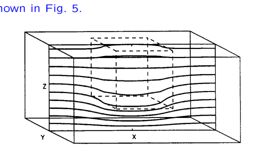

Fig. 4: See text for details.

Figure 5 shows that the multi-grid methods with 3 and 4 levels converge smooth and fastest. However, since the computational costs of one multi-grid iteration is 2 to 7 times that of a single Λ-iteration, the orthomin method (Klein et al., 1989) is more efficient. This is because on this massively parallel machine the CPU-time needed for a formal solution scales with N, compared with N^3 on a scalar machine. Correspondingly, the computational costs on the coarser multi-grid levels reduce just as 2^{-l} instead of 2^{-3l} making the coarse grid iterations relatively expensive. Also the load balance of the processor units is difficult to maintain for the multi-grid method. Väth (1994) concludes that multi-grid radiation transfer is *not well suited* for SIMD machines.

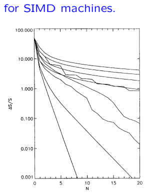

Fig. 5: Maximum relative error of the source function as function of iteration steps. From top to bottom (worst to best): Λ-iteration, local ALI, non-local ALI, 2 level MGM, Ng acceleration, local orthomin, non-local orthomin, level 3 MGM, level 4 MGM.

Multilevel atoms in multidimensional space: Non-linear multigrid radiation transfer

Fabiani Bendicho *et al.* (1997) apply multi-grid radiation transfer to the multilevel non-LTE problem in one and two spatial dimensions. The population equation is given by

$$\mathcal{L}_l \cdot \mathbf{n}_l = \mathbf{f}_l\,, \tag{14}$$

where $\mathcal{L}_l$ is a block-diagonal matrix formed by NP_l submatrices, each $\mathrm{NL} \times \mathrm{NL}$ in size. NP_l is the number of depth points on grid level l, NL the number of atomic levels. $\mathcal{L}_l$ depends on the radiative rates, which are computed via the solution of the radiation transfer equation, which in turn depends on $\mathbf{n}_l$.

Fabiani Bendicho *et al.* (1997) use a Gauss-Seidel iteration scheme, which they call MUGA, for solving this non-linear problem. They emphasize the excellent smoothing capability of MUGA, making it an ideal choice for the smoothing iterations in the multi-grid method.

The *non-linear multi-grid method* differs from the linear case in that the coarse-grid equation solves for the occupation numbers, $\mathbf{n}_{l-1}$, directly and not just for the correction $\Delta\mathbf{n}_{l-1}$, and in that the restriction operator is not only applied to the residual but to the current estimate $\mathbf{n}_l^{\mathrm{old}}$ as well.

Figure 6 shows the *contraction number* as a function of grid spacing (in km) of three schemes for the test case of a 5-level Ca II atom in a one-dimensional (solar) atmosphere. The contraction number is basically the factor by which the error is reduced per iteration. MALI stands for the multi-level accelerated Λ-iteration, MUGA for the multi-level Gauss-Seidel method, and TGM for the two-grid method.

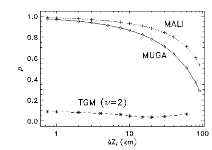

Fig. 6: See text for details.

Figure 6 demonstrates the known and attractive property of multi-grid methods that their convergence rates are virtually *independent on grid resolution*, contrary to operator splitting methods that deteriorate when resolution is increased. Second, the contraction number for the two-grid method is $\rho < 0.1$, meaning that with each iteration the error is reduced by one order of magnitude.

Figure 7 demonstrates the convergence in terms of CPU-time on a scalar computer of the multi-grid method for the 5-level Ca II atom in a two-dimensional atmosphere with a sinusoidal variation of wavelength P. C_e is the relative error with respect to the accurate solution, Δz the grid spacing in km. Note that *within at most 3 iterations*, the multi-grid method has converged to $C_e < 10^{-5}$.

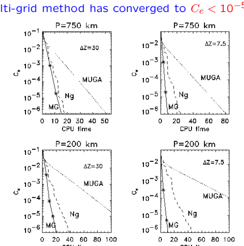

Fig. 7: See text for details.

Conclusions

Multi-grid radiation transfer is an efficient method for solving a variety of radiation transfer problems. It is efficient for problems of multiple spatial dimensions on scalar computers. This advantage is lost on massively parallel machines in which the computational grid can be directly mapped onto the processor array. Contrary to operator splitting methods, the convergence rate of the multi-grid method does not deteriorate with increasing spatial resolution of the computational grid. It is therefore well suited for high resolution problems, while performance at low resolution is not better than the best operator splitting methods.

Because of the variety of existing multi-grid algorithms, there is still ample room for improvements of the few multi-grid radiation-transfer calculations that have been performed so far.

References

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